Limits

Computing Limits Using the Laws of Limits

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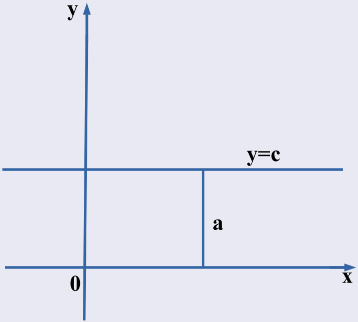
March 28, 2020

LAW - 1: Limit of a Constant Function $f(x) = c$

If c is a real number, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} c = c$$

Graphical Description



Example

$$\lim_{x\to 2} 5 = 5$$

Example

$$\lim_{x\to -1} 3 = 3$$

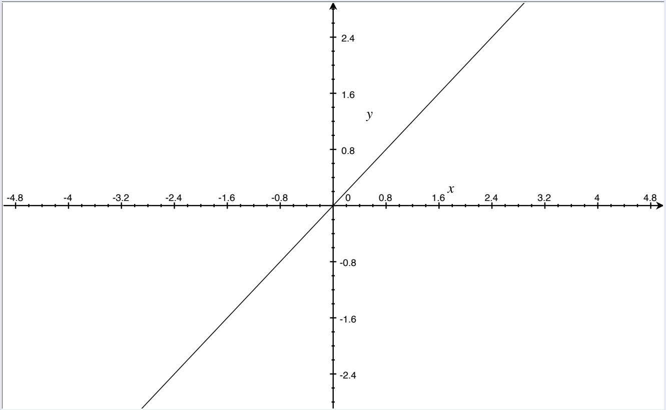
Example

$$\lim_{x\to 0} 2\pi = 2\pi$$

LAW - 2: Limit of the Identity Function $f(x) = x$

$$\lim_{x \to a} f(x) = \lim_{x \to a} x = a$$

Graphical Description



Example

$$\lim_{x \to 4} x = 4$$

Example

$$\lim_{x \to 0} x = 0$$

Example

$$\lim_{x \to -\pi} x = -\pi$$

LIMIT LAWS

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then we have the following laws.

LAW - 3: Sum Law

$$\lim_{x \to a} f(x) \pm g(x) = L \pm M$$

LAW - 4: Product Law

$$\lim_{x \to a} [f(x)g(x)] = LM$$

LAW - 5: Constant Multiple Law

$\lim_{x \to a} [cf(x)] = cL$, for every c.

LAW - 6: Quotient Law

$\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, provided $M \neq 0$.

LIMIT LAWS

LAW - 6: Root Law

$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$, provided that n is a positive integer, and $L > 0$ if n is even.

Remark

Sum Law and the Product Law are stated for two functions, they are also valid for any finite number of functions. That is if $\lim_{x \to a} f_1(x) = L_1, \lim_{x \to a} f_2(x) = L_2, \ldots, \lim_{x \to a} f_n(x) = L_n$, then we have the following

$$\lim_{x \to a}[f_1(x) + f_2(x) + \cdots + f_n(x)] = L_1 + L_2 + \cdots + L_n$$

$$\lim_{x \to a}[f_1(x)f_2(x) \cdots f_n(x)] = L_1 L_2 \cdots L_n \quad (1)$$

Remark

If we take $f_1(x) = f_2(x) = \cdots = f_n(x) = f(x)$, then Equation (1) gives the following result for powers of f.

LAW 8:

If n is a positive integer and $\lim_{x \to a} f(x) = L$, then $\lim_{x \to a} [f(x)]^n = L^n$.

Remark

If we take $f(x) = x$, then Equation (1) and Law (8) give the following result.

LAW 9:

$\lim_{x \to a} x^n = a^n$ where n is a positive integer

Example

Find $\lim_{x \to 2} 2x^3 - 4x^2 + 3$

$$\begin{aligned}
\lim_{x \to 2} 2x^3 - 4x^2 + 3 &= \lim_{x \to 2} 2x^3 - \lim_{x \to 2} 4x^2 + \lim_{x \to 2} 3 && \textbf{Law 3} \\
&= 2\lim_{x \to 2} x^3 - 4\lim_{x \to 2} x^2 + \lim_{x \to 2} 3 && \textbf{Law 5} \\
&= 2(2)^3 - 4(2)^2 + 3 && \textbf{Law 9} \\
&= 3.
\end{aligned}$$

LAW 10: Limits of Polynomial Functions

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial function, then $\lim_{x\to a} p(x) = p(a)$.

Proof by repeated applycation of the (generalized) sum law and the constant multiple law.

$$\begin{aligned}\lim_{x\to a} p(x) &= \lim_{x\to a} a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\ &= a_n(\lim_{x\to a} x^n) + a_{n-1}(\lim_{x\to a} x^{n-1}) + \cdots + \lim_{x\to a} a_0\end{aligned}$$

By laws 1, 2, and 9,

$$\begin{aligned}\lim_{x\to a} p(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots a_0 \\ &= p(a)\end{aligned}$$

□

LAW - 11: Limits of Rational Functions

If f is a rational function defined by $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial functions and $Q(a) \neq 0$, then $\lim_{x \to a} f(x) = f(a) = \dfrac{P(a)}{Q(a)}$

Proof.

By applying LAW 10 and using the quotient law ☐

1. If the numerator does not approach 0 but the denominator does, then the limit of the quotient does not exist.
2. A function whose limit at a can be found by evaluating it at a is said to be continuous at a.

Theorem (Limits of Trigonometric Functions)

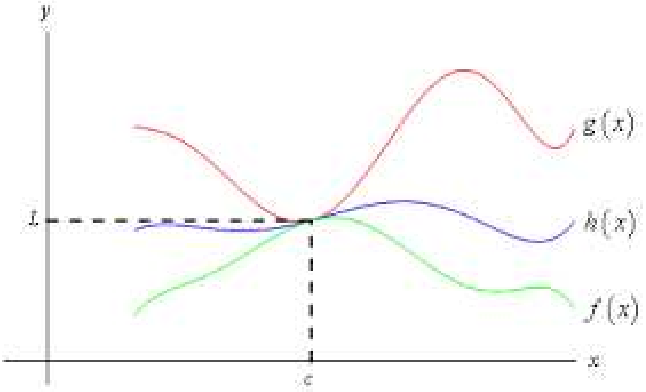
Let a be a number in the domain of the given trigonometric function. Then

- $\lim_{x \to a} \sin x = \sin a$
- $\lim_{x \to a} \cos x = \cos a$
- $\lim_{x \to a} \tan x = \tan a$
- $\lim_{x \to a} \cot x = \cot a$
- $\lim_{x \to a} \sec x = \sec a$
- $\lim_{x \to a} \csc x = \csc a$

The above techniques that we have developed so far do not work in all situations. For example, $\lim_{x \to 0} x^2 \sin(\frac{1}{x})$. Hence in such cases we apply **Squeeze Theorem**

Theorem (The Squeeze Theorem)

Suppose that $f(x) \leq g(x) \leq h(x)$, *for all* x *in an open interval containing* a, *except possibly at* a, *and* $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$ *Then* $\lim_{x \to a} g(x) = L$.



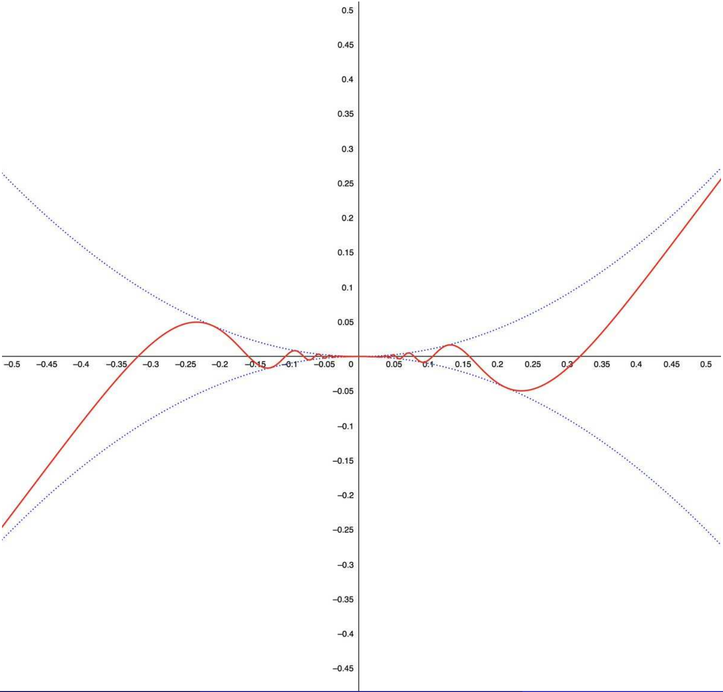
Example

Find $\lim_{x \to a} x^2 \sin(\frac{1}{x})$

Since $-1 \leq \sin t \leq 1$ for every real number t, we have for every $x \neq 0$, $-1 \leq \sin \frac{1}{x} \leq 1$. Therefore for all

$$x \neq 0, -x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$$

By choosing $f(x) = -x^2, g(x) = x^2 \sin \frac{1}{x}, h(x) = x^2$, we have $f(x) \leq g(x) \leq h(x)$. Since $\lim_{x \to 0} f(x) = \lim_{x \to 0}(-x^2) = 0 = \lim_{x \to 0} h(x) = \lim_{x \to 0}(x^2)$, By **Squeeze theorem** we have $\lim_{x \to 0} g(x) = 0$.

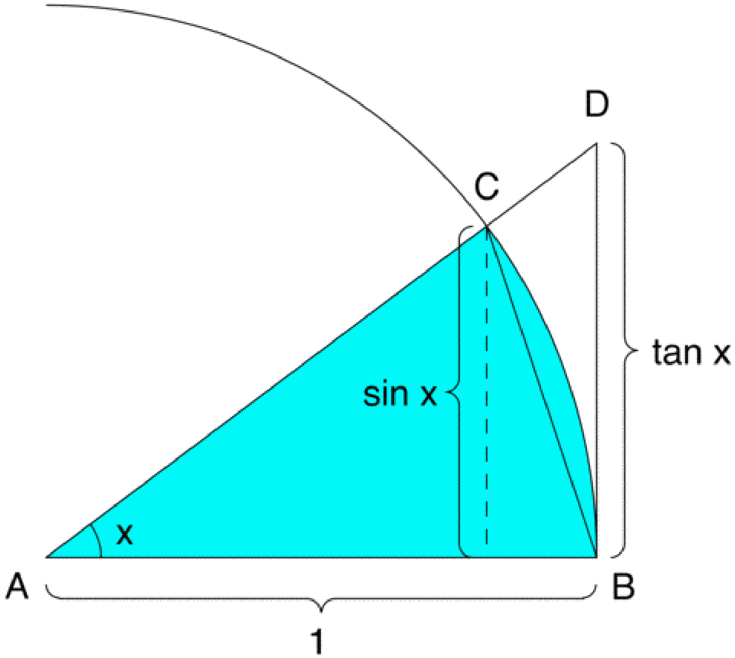

0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
−0.5 −0.45 −0.4 −0.35 −0.3 −0.25 −0.2 −0.15 −0.1 −0.05 0 0.05 0.1 0.15 0.2 0.25 0.3 0.35 0.4 0.45 0.5
−0.05
−0.1
−0.15
−0.2
−0.25
−0.3
−0.35
−0.4
−0.45

Theorem

Suppose that $f(x) \leq g(x)$ *for all* x *in an open interval containing* a*, except possibly at* a*, and* $\lim_{x \to a} f(x) = L$ *and* $\lim_{x \to a} g(x) = M$*. Then* $L \leq M$*.*

Theorem

$$\lim_{x \to 0} \frac{\sin x}{x} = 0$$


D
C
tan x
sin x
x
A
B
1

Proof.

Suppose $0 < x < \frac{\pi}{2}$. From the Figrue just shown, we have the following.

$$\text{Area of } \Delta ABC = \frac{1}{2}(1)\sin x = \frac{1}{2}\sin x \qquad \frac{1}{2}\textit{base.height}$$

$$\text{Area of sector} ABC = \frac{1}{2}(1)^2 x = \frac{1}{2}x \qquad \frac{1}{2}r^2 x$$

$$\text{Area of } \Delta ABD = \frac{1}{2}(1)\tan x = \frac{1}{2}\tan x$$

Since $0 <$ Area of $\Delta ABC <$ Area of sector$ABC <$ Area of ΔABD, we have

$$0 < \frac{1}{2}\sin x < \frac{1}{2}x < \frac{1}{2}\tan x \tag{2}$$

. We know that in the interval $0 < x < \frac{\pi}{2}$, we have $\sin x > 0, \cos x > 0$
Multiplying the equation 2 throughout by $\frac{2}{\sin x}$, we get

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

Proof.

That is

$$\cos x < \frac{\sin x}{x} < 1 \tag{3}$$

if $-\frac{\pi}{2} < x < 0$, then $0 < -x < \frac{\pi}{2}$ and Inequality (3) gives

$$\cos(-x) < \frac{\sin(-x)}{(-x)} < 1$$

$$\cos x < \frac{\sin x}{x} < 1 \text{ which is same as Inequality (3)}$$

Therefore the inequality (3) holds whenever x lies in $(-\frac{\pi}{2}, 0)$ and $(0, \frac{\pi}{2})$.
Now choose $f(x) = \cos x, g(x) = \frac{\sin x}{x}, h(x) = 1$, and
$\lim_{x\to 0} f(x) = \lim_{x\to 0} \cos x = 1$ and $\lim_{x\to 0} h(x) = \lim_{x\to 0} 1 = 1$, by **Squeeze theorem** we have $\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1$ □

Theorem

$$\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$$

Proof.

We know that $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$, by choosing $2\theta = x$, we have $1 - \cos x = 2\sin^2 \frac{x}{2}$. Then

$$\begin{aligned}
\lim_{x \to 0} \frac{\cos x - 1}{x} &= \lim_{x \to 0} \left(\frac{-2\sin^2(\frac{x}{2})}{x} \right) \\
&= \lim_{x \to 0} (-\sin \frac{x}{2}) \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right) \\
&= -\lim_{x \to 0} (\sin \frac{x}{2}) \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right) \\
&= 0.1 \\
&= 0.
\end{aligned}$$